Moment Explosions and Asymptotic Behavior of the Stock Price Distribution in Heston Model

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Abstract

We study the asymptotic behavior of the cumulative distribution an arbitrary combination of the CIR process and its time average and the stock price in Heston model. We find an asymptotic formulas for the cumulative distributions of these distributions and obtain a sharp asymptotic formulas for the implied volatility in the Heston model[1].

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1 Introduction

The CIR process has been proposed by Cox, Ingersoll and Ross [1] to model the evolution of interest rates. It is defined as the unique solution of the following stochastic differential equation:

$$dV_t = (a - bV_t)dt + \sigma\sqrt{V_t}dW_t, \quad V_0 = v, \tag{1.1}$$

where $a,\ \sigma,\ v\ \geq 0$ and $b \in \mathbb{R}$ (see [6] for the existence and uniqueness of the SDE). The density transition V_t is known and is given by (cf. [7])

$$P_t(v, z) = \frac{e^{bt}}{2c_t}\left(\frac{ze^{bt}}{v}\right)^{\frac{\nu-1}{2}} \exp\left(-\frac{v + ze^{bt}}{2c_t}\right) I_\nu\left(\frac{1}{c_t}\sqrt{vze^{bt}}\right), \tag{1.2}$$

[1] I would like to thank Prof Damien Lamberton for many useful discussions.

where $c_t = \frac{(e^{bt}-1)\sigma^2}{4b}$, $\nu = 2a/\sigma^2$ and I_ν is the modified Bessel function of order ν. We set

$$I_t = \int_0^t V_u du \tag{1.3}$$

Gulisashvili and Stein [4] (cf Lemma 6.3 and Remark 8.1) show that I_t admits a distribution density (see also Dufresne [3] for a similar result). A sharp asymptotic formulas for the distribution density of I_t is obtained in [4] (cf Theorem 2.4) as

$$p_t^I(y) = A_t\, e^{-C_t y + B_t\sqrt{y}}\, y^{\frac{\nu}{2}-\frac{3}{4}} \left(1 + \mathcal{O}(y^{-\frac{1}{2}})\right) \tag{1.4}$$

where A, B and C are positive constants. The integral of V arises in both volatility and interest rate models. For example, the CIR process is used to model the instantaneous variance of underlying asset in the Heston model [5], where the dynamics of the log-price of the underlying asset is given by the SDE

$$dX_t \quad = \quad -\frac{1}{2}V_t + \sqrt{V_t}\left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right) \tag{1.5}$$

whith $\rho \in]-1,1[$. Heston's model is certainly the most popular among stochastic volatility models. Its popularity is mainly due to the availability of semi-explicit formulas for the prices of European options on the underlying asset using the fact that the Laplace transform of X_t is given explicitly. However, there is no explicit formulas for the distribution density of X_t. In the particular case where $\rho = 0$, a sharp asymptotic formulas for the distribution density is given in [4] as

$$p_t^{X,\rho=0}(x) = A_1 e^{-A_3 x + A_2\sqrt{x}} x^{\frac{a}{\sigma^2}-\frac{3}{4}} \left(1 + \mathcal{O}(x^{-\frac{1}{4}}.)\right) \tag{1.6}$$

where A_1, A_2 and A_3 are positive constants. In general case, Dragulescu and Yakovenko [2] show that the density of X_t behaves as $e^{-p_+ x}$ for $x \to +\infty$ (and $e^{-p_- x}$ for $x \to -\infty$). In particular, the moment generating function of X_t becomes infinite ('explodes') after the 'critical moment' $\mu^* = p_+$. This moment explosion phenomenon has also been linked to the wing behavior of the implied volatility, via Lee's celebrated moment formula [8].

The purpose of the paper is to connect the tail behavior of some real-valued random variable to the behavior of the moment generating function near the 'critical moment'. Our main result is to show that if Z is real-value random variable for which $\mu \longmapsto \mathbb{E}\, e^{\mu Z}$ explodes at some critical moment μ^* and behaves in a certain way near μ^* (see Assumption 3.1), then one can obtain a sharp asymptotic formulas for the cumulative distribution of Z, where the "secondary-terms" are given in lim sup statement. We show that this result can be applied to random-variables of type $\lambda_1 V_t + \lambda_2 I_t$ as well as X_t. As an application of our results, sharp asymptotic formula for

the implied volatility in the Heston model is obtained.

This paper is organized as follows: In section 2 we give the moment generating function of an arbitrary combination of V and I. In section 3 we give a large deviation result linking the moment explosion and the tail behavior. The section 4 gives the tail behavior of X and the wing behavior of the implied volatility. We give some proof in the appendix.

2 Moment generating function of a combination of V and I

To compute the moment generating function of some combination of V and I, we use the additivity property of V with respect to (a, v). Indeed, if we denote by $V_t^{v,a}$the solution of (1.1) with initial condition v, then for any (v_1, v_2),$(a_1, a_2) \in \mathbb{R}_+^2$, the process $V^{v_1+v_2,a_1+a_2}$ has the same low as $(V_t^{v_1,a_1} + \hat{V}_t^{v_2,a_2})$ where $\hat{V}^{v,a}$ is the unique solution of (1.1) in which W is replaced by an independent Brownian motion. Indeed, by setting $V_t^1 = V_t^{v_1,a_1}$, $\hat{V}_t^2 = \hat{V}_t^{v_2,a_2}$ and $V_t = V_t^1 + \hat{V}_t^2$, we have

$$\begin{aligned} V_t &= v_1 + v_2 + \int_0^t [(a_1 + a_2) - bV_s]\, ds + \sigma \int_0^t \left(\sqrt{V_s^1} dW_s + \sqrt{\hat{V}_s^2} d\hat{W}_s \right) \\ &= v_1 + v_2 + \int_0^t [(a_1 + a_2) - bV_s]\, ds + \sigma \int_0^t \sqrt{V_s} dB_s, \end{aligned}$$

where B is Brownian motion defined as

$$dB_s = \frac{\sqrt{V_s^1}}{\sqrt{V_s^1 + \hat{V}_s^2}} dW_s + \frac{\sqrt{\hat{V}_s^2}}{\sqrt{V_s^1 + \hat{V}_s^2}} d\hat{W}_s.$$

It follows from that if we set, for λ_1 and $\lambda_2 \in \mathbb{R}$,,

$$F_{\lambda_1,\lambda_2}^a(t, v) = \mathbb{E}\ e^{\lambda_1 V_t^{v,a} + \lambda_2 I_t},$$

we have $F_{\lambda_1,\lambda_2}^{a_1+a_2}(t, v_1 + v_2) = F_{\lambda_1,\lambda_2}^{a_1}(t, v_1) F_{\lambda_1,\lambda_2}^{a_2}(t, v_2)$. So the function $F_{\lambda_1,\lambda_2}^a$ takes the form

$$F_{\lambda_1,\lambda_2}^a(t, v) = e^{a\varphi_{\lambda_1,\lambda_2}(t) + v\psi_{\lambda_1,\lambda_2}(t)}.$$

The functions φ and ψ are characterized by the following result, whose proof can be found in the appendix.

Theorem 2.1. *For $\lambda_1,\ \lambda_2\ \in\mathbb{R}$, denote by $\psi_{\lambda_1,\lambda_2}$ the maximal solution of*

$$\begin{cases} \psi'_{\lambda_1,\lambda_2}(t) = \dfrac{\sigma^2}{2}\left(\psi^2_{\lambda_1,\lambda_2}(t) - 2\dfrac{b}{\sigma^2}\psi_{\lambda_1,\lambda_2}(t) + 2\dfrac{\lambda_2}{\sigma^2}\right), \\ \psi_{\lambda_1,\lambda_2}(0) = \lambda_1, \end{cases} \tag{2.1}$$

defined over $[0, t^*_{\lambda_1,\lambda_2}[$ and denote by $\varphi_{\lambda_1,\lambda_2}(t) := \int_0^t \psi_{\lambda_1,\lambda_2}(u)du$. Then, for any $T < t^*_{\lambda_1,\lambda_2}$, we have

$$\mathbb{E}\, e^{\lambda_1 V_T^{v,a} + \lambda_2 I_T} \ = \ e^{a\varphi_{\lambda_1,\lambda_2}(T) + v\psi_{\lambda_1,\lambda_2}(T)} \tag{2.2}$$

The function $F^a_{\lambda_1,\lambda_2}$ is defined over $[0, t^*_{\lambda_1,\lambda_2}[$ s.t $\lim_{t\to t^*_{\lambda_1,\lambda_2}} F^a_{\lambda_1,\lambda_2}(t,v) = +\infty$. For $\lambda_1,\ \lambda_2\ \in \mathbb{R}$, denote by $t^*_{\lambda_1,\lambda_2}(\mu) = t^*_{\mu\lambda_1,\mu\lambda_2}$. Denote also by $\mu^*_-(t) = -\sup\left\{\mu < 0 :\ \ t^*_{\lambda_1,\lambda_2}(\mu) = t\right\}$ and $\mu^*_+(t) = \inf\left\{\mu > 0 :\ \ t^*_{\lambda_1,\lambda_2}(\mu) = t\right\}$ (with $\sup\Phi = -\infty$ and $\inf\Phi = +\infty$). So the moment generating function of $\lambda_1 V_t + \lambda_2 I_t$ is defined over $]-\mu^*_-(t), \mu^*_+(t)[$ and if $\mu^*_-(t) < 0$ (resp $\mu^*_+(t) < 0$) then $\lim_{\mu\to -\mu^*_-(t)} \mathbb{E}\, e^{\mu\lambda_1 V_t + \mu\lambda_2 I_t} = +\infty$ (resp $\lim_{\mu\to\mu^*_+(t)} \mathbb{E}\, e^{\mu\lambda_1 V_t+\mu\lambda_2 I_t} = +\infty$).

Proposition 2.2. *Let $t > 0$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ so that $\max(\lambda_1, \lambda_2) > 0$. Then $\mu^*_+(t) < +\infty$. Furthermore, $\exists \omega^+_t > 0$ s.t $\mu \longmapsto \left(\ln\left(\mathbb{E}\, e^{\mu\lambda_1 V_t + \mu\lambda_2\int_0^t V_u du}\right) - \left(\frac{\omega^+_t}{\mu^*_+(t)-\mu} - \frac{2a}{\sigma^2}\ln\frac{1}{\mu^*_+(t)-\mu}\right)\right)$ is bounded near $\mu^*_+(t)$.*

3 Moment explosion and asymptotic behavior of the cumulative distribution

In this section we present our main result based on the moment explosion of some random variable Z

Assumption 3.1. *There exist $\mu^*, \omega \in \mathbb{R}^*_+$ and $\nu \in \mathbb{R}$ so that $\mathbb{E}\, e^{\mu Z} < +\infty,\ \forall\ \mu \in [0, \mu^*[$ and $\left(\ln\left(\mathbb{E}\, e^{\mu Z}\right) - \Lambda(\mu)\right)$ is bounded near μ^*, where $\Lambda(\mu) = \frac{\omega}{\mu^*-\mu} + \nu\ln\frac{1}{\mu^*-\mu}$.*

We next present our main result concerning the asymptotic behavior of the cumulative distribution of a random variable Z for which Assumption 3.1 holds.

Theorem 3.2. *Let Assumption 3.1 hold. Then we have*

$$\lim_{R\to\infty} \frac{\ln\mathbb{P}(Z > R) + \mu^* R}{R^\alpha} = 0, \quad \forall\alpha > \frac{3}{4}, \tag{3.1}$$

$$\limsup_{R\to\infty} \frac{\ln\mathbb{P}(Z > R) + \mu^* R - 2\sqrt{\omega R}}{\ln(R)} \in \left[\frac{\nu}{2} - \frac{3}{4}, \frac{\nu}{2}\right] \tag{3.2}$$

Furthermore, if $\lim_{R\to\infty} \frac{\ln \mathbb{P}(Z>R)+\mu^* R - 2\sqrt{\omega R}}{\ln(R)} = \nu^*$, *then*

$$\nu^* = \frac{\nu}{2} - \frac{3}{4} \tag{3.3}$$

Corollary 3.3. *For any* $t > 0$ *and* $\lambda_1, \lambda_2 \in \mathbb{R}$ *so that* $\max(\lambda_1, \lambda_2) > 0$, *(3.1), (3.2) and (3.3) hold for* $Z := \lambda_1 V_t + \lambda_2 I_t$.

4 Asymptotic behavior of the stock price distribution in Heston model

The analysis relies on the explicit calculation of the moment generating function of X. Let's define the quantity

$$F_p(t) = \mathbb{E}\, e^{pX_t}, \quad p \in \mathbb{R}. \tag{4.1}$$

We have for all $p \in \mathbb{R}$

$$\begin{aligned}
F_p(t) &= \mathbb{E}\, e^{-\frac{p}{2}\int_0^t V_s ds + p\rho \int_0^t \sqrt{V_s} dW_s^1 + p\sqrt{1-\rho^2}\int_0^t \sqrt{V_s} dW_s^2} \\
&= \mathbb{E}\left\{ e^{\frac{p^2(1-\rho^2)-p}{2}\int_0^t V_s ds + p\rho \int_0^t \sqrt{V_s} dW_s^1} \left[\mathbb{E}\, e^{p\sqrt{1-\rho^2}\int_0^t \sqrt{V_s} dW_s^2 - \frac{p^2(1-\rho^2)}{2}\int_0^t V_s ds} \middle| (W_s^1)_{s\le t} \right] \right\} \\
&= \mathbb{E}\left[e^{p\rho \int_0^t \sqrt{V_s} dW_s^1 - \frac{p^2\rho^2}{2}\int_0^t V_s ds}\, e^{\frac{p^2-p}{2}\int_0^t V_s ds} \right] \\
&= \mathbb{E}^{\mathbb{Q}}\, e^{\frac{p^2-p}{2}\int_0^t V_s ds}
\end{aligned}$$

where we used the law of iterated conditional expectation and the fact that V_s is measurable with respect to W^1. The last inequality is a consequence of Girsanov theorem, where under $\mathbb{Q}$, the process V satisfies the stochastic differential equation

$$dV_t = (a - (b - \rho\sigma p)V_t)\, dt + \sigma\sqrt{V_t} dW_t^{1,\mathbb{Q}} \tag{4.2}$$

with $\mathbb{Q}-$ Brownian motion $W^{1,\mathbb{Q}}$. We are the reduced to the calculation of the moment generating function of the time average of the CIR process V under $\mathbb{Q}$. It follows that $F_p(t) = e^{a\varphi_p(t)+v\psi_p(t)}$, where $\varphi_p(t) = \int_0^t \psi_p(s)ds$ and ψ_p is given by theorem 2.1.

Theorem 4.1. *For any* $t > 0$, *Assumption 3.1 holds for* X_t *as well as for* $-X_t$. *In particular, (3.1), (3.2) and (3.3) hold for* X_t *and* $(-X_t)$.

Asymptotic behavior of the implied volatility We can also analyze the asymptotic behaviour for the implied volatility. Recall that the implied volatility $\sigma_t = \sigma_t(k)$ of a call option

$S_t = e^{X_t}$ with strike $K = e^k$, and maturity t is determined from the relation:

$$\mathbb{E}\left(e^{X_t} - e^k\right)_+ = C_{BS}(t, k, \sigma_t) := N(d_1(t, k, \sigma_t)) - e^k N(d_2(t, k, \sigma_t)) \tag{4.3}$$

where

$$d_1(t, k, \sigma) = \frac{-k + \frac{1}{2}\sigma^2 t}{\sqrt{t}\sigma} \quad \text{and} \quad d_2(t, k, \sigma_t) = d_1(t, k, \sigma_t) - \sqrt{t}\sigma.$$

and N is the cdf of the normal law $\mathcal{N}(0,1)$. The asymptotic behavior of the implied volatility is linked to the tail behavior of the distribution of the stock price process. Here we get the following result

Theorem 4.2. *for any $t > 0$, we have*

$$\lim_{k\to+\infty} \frac{\sigma_t(\pm k) - \beta_1^{\pm}\sqrt{k}}{k^{\gamma}} = 0, \quad \forall \gamma > \frac{1}{4}, \tag{4.4}$$

$$\limsup_{k\to+\infty} \frac{\sqrt{k}}{\beta_3^{\pm}\ \ln(k)} \left(\sigma_t(\pm k) - (\beta_1^{\pm}\sqrt{k} + \beta_2^{\pm})\right) \in \left[\frac{a}{\sigma^2} - \frac{1}{4}, \frac{a}{\sigma^2} + \frac{1}{2}\right] \tag{4.5}$$

And if (3.3) holds for $\pm X_t$, then

$$\lim_{k\to+\infty} \frac{\sqrt{k}}{\beta_3^{\pm}\ \ln(k)} \left(\sigma_t(\pm k) - (\beta_1^{\pm}\sqrt{k} + \beta_2^{\pm})\right) = \frac{a}{\sigma^2} - \frac{1}{4}. \tag{4.6}$$

Whith $\beta_1^{\pm} = \frac{\sqrt{2}}{\sqrt{t}}\left(\sqrt{p_{\pm}^X} - \sqrt{p_{\pm}^X - 1}\right)$, $\beta_2^{\pm} = \frac{\sqrt{2\omega_{\pm}}}{\sqrt{t}}\left(\frac{1}{\sqrt{p_{\pm}^X - 1}} - \frac{1}{\sqrt{p_{\pm}^X}}\right)$, $\beta_3^{\pm} = \frac{1}{\sqrt{2t}}\left(\frac{1}{\sqrt{p_{\pm}^X - 1}} - \frac{1}{\sqrt{p_{\pm}^X}}\right)$, $p_+ = \mu_+^X(t)$ *and* $p_- = \mu_-^X(t) + 1$, *where* $\mu_{\pm}^X$ *and* $\omega_{\pm}$ *are given by (E.4), (E.5), (E.7), (E.10) and (E.11).*

A Proof of Theorem 2.1

Denote by

$$F^a_{\lambda_1,\lambda_2}(t, v) = e^{a\varphi_{\lambda_1,\lambda_2}(t) + v\psi_{\lambda_1,\lambda_2}(t)}.$$

The function $F^a_{\lambda_1,\lambda_2}$ is the maximal solution of

$$\begin{cases} \dfrac{\partial F^a_{\lambda_1,\lambda_2}}{\partial t}(t, v) = \dfrac{\sigma^2}{2} v \dfrac{\partial^2 F^a_{\lambda_1,\lambda_2}}{\partial v^2}(t, v) + (a - bv)\dfrac{\partial F^a_{\lambda_1,\lambda_2}}{\partial v}(t, v) + \lambda_2 v F^a_{\lambda_1,\lambda_2}(t, v), \quad 0 < t \le T, \\ \\ F^a_{\lambda_1,\lambda_2}(0, v) = e^{\lambda_1 v}. \end{cases}$$

Indeed, this is equivalent to $\varphi_{\lambda_1,\lambda_2}(0) = 0$, $\psi_{\lambda_1,\lambda_2}(0) = \lambda_1$ and

$$\begin{aligned} a\varphi'_{\lambda_1,\lambda_2}(t) + v\psi'_{\lambda_1,\lambda_2}(t) &= \frac{\sigma^2}{2} v\psi^2_{\lambda_1,\lambda_2}(t) + (a - bv)\psi_{\lambda_1,\lambda_2}(t) + \lambda_2 v \\ &= a\psi_{\lambda_1,\lambda_2}(t) + v\left(\frac{\sigma^2}{2}\psi^2_{\lambda_1,\lambda_2}(t) - b\psi(t) + \lambda_2\right). \end{aligned}$$

This is equivalent to $\varphi'_{\lambda_1,\lambda_2}(t) = \psi_{\lambda_1,\lambda_2}(t)$ and

$$\psi'_{\lambda_1,\lambda_2}(t) = \frac{\sigma^2}{2}\left(\psi^2_{\lambda_1,\lambda_2}(t) - 2\frac{b}{\sigma^2}\psi_{\lambda_1,\lambda_2}(t) + 2\frac{\lambda_2}{\sigma^2}\right).$$

So for $T < t^*_a(\lambda_1, \lambda_2)$, the process $\left(e^{\lambda_2 \int_0^s V_u^{v,a} du} F^a_{\lambda_1,\lambda_2}(T - s, V_s^{v,a})\right)_{s \leq T}$ is a positive local martingale. It is therefore a super-martingale. In particular, for any stopping time $\tau \in \mathcal{T}_{0,T}$, we have

$$E\, e^{\lambda_2 \int_0^\tau V_u^{v,a} du} F^a_{\lambda_1,\lambda_2}(T - \tau, V_\tau^{v,a}) \leq F^a_{\lambda_1,\lambda_2}(T, V_0^{v,a})$$

It follows that

$$\sup_{\tau \in \mathcal{T}_{0,T}} E\, e^{\lambda_2 \int_0^\tau V_u^{v,a} du + a\varphi_{\lambda_1,\lambda_2}(T-\tau) + V_\tau^{v,a}\psi_{\lambda_1,\lambda_2}(T-\tau)} \leq e^{a\varphi_{\lambda_1,\lambda_2}(T) + v\psi_{\lambda_1,\lambda_2}(T)} \tag{A.1}$$

We will show that there exists $p > 1$ such that

$$\sup_{\tau \in \mathcal{T}_{0,T}} E\, e^{p\left(\lambda_2 \int_0^\tau V_u^{v,a} du + a\varphi_{\lambda_1,\lambda_2}(T-\tau) + V_\tau^{v,a}\psi_{\lambda_1,\lambda_2}(T-\tau)\right)} < +\infty$$

Note that $\psi_{\lambda_1,\lambda_2}$ is increasing with respect to λ_1 and λ_2. This means that if $\lambda_1 < \bar{\lambda}_1$ (resp $\lambda_2 < \bar{\lambda}_2$), we have $\psi_{\lambda_1,\lambda_2}(t) \leq \psi_{\bar{\lambda}_1,\lambda_2}(t)$, for every $t < t^*(\bar{\lambda}_1, \lambda_2)$ (resp $\psi_{\lambda_1,\lambda_2}(t) \leq \psi_{\lambda_1,\bar{\lambda}_2}(t)$, $\forall t < t^*(\lambda_1, \bar{\lambda}_2)$). Indeed, if we set $f = \psi_{\bar{\lambda}_1,\lambda_2} - \psi_{\lambda_1,\lambda_2}$ and $g = \psi_{\lambda_1,\bar{\lambda}_2} - \psi_{\lambda_1,\lambda_2}$, we have

$$\begin{aligned} f(0) &= \bar{\lambda}_1 - \lambda_2 > 0, \\ f'(t) &= \frac{\sigma^2}{2}\left(\psi^2_{\bar{\lambda}_1,\lambda_2}(t) - \psi^2_{\lambda_1,\lambda_2} - \frac{2b}{\sigma^2} f(t)\right) \\ &= \frac{\sigma^2}{2} f(t)\left(\psi_{\bar{\lambda}_1,\lambda_2}(t) + \psi_{\lambda_1,\lambda_2} - \frac{2b}{\sigma^2}\right) \end{aligned}$$

and

$$
\begin{aligned}
g(0) &= 0, \\
g'(t) &= \frac{\sigma^2}{2}\left(\psi^2_{\bar\lambda_1,\lambda_2}(t) - \psi^2_{\lambda_1,\lambda_2} - \frac{2b}{\sigma^2} g(t) + 2\frac{\bar\lambda_2 - \lambda_2}{\sigma^2}\right) \\
&\geq \frac{\sigma^2}{2} g(t)\left(\psi_{\bar\lambda_1,\lambda_2}(t) + \psi_{\lambda_1,\lambda_2} - \frac{2b}{\sigma^2}\right).
\end{aligned}
$$

It follows that f and g satisfy $(fe^A)'(t) = 0$, with $f(0) > 0$ and $(ge^A)'(t) \geq 0$ with $g(0) = 0$, where $A'(t) = \frac{\sigma^2}{2}\left(\psi_{\bar\lambda_1,\lambda_2}(t) + \psi_{\lambda_1,\lambda_2} - \frac{2b}{\sigma^2}\right)$. Thus $f \geq 0$ and $g \geq 0$.

It follows that

$$\psi_{\bar\lambda_1,\lambda_2}(t) - \psi_{\lambda_1,\lambda_2}(t) = (\bar\lambda_1 - \lambda_1)\, e^{bt - \int_0^t (\psi_{\bar\lambda_1,\lambda_2} + \psi_{\lambda_1,\lambda_2})(s)ds}, \quad \forall t < t^*(\bar\lambda_1, \lambda_2)$$

So for every $T < t^*_{\lambda_1,\lambda_2}$ and for $\epsilon > 0$ small enough, there exists $\lambda_1^\epsilon > \lambda_1$ such that

$$(1+\epsilon)\psi_{\lambda_1,\lambda_2}(t) \leq \psi_{\lambda_1^\epsilon,\lambda_2}(t), \quad \forall t \leq T.$$

We deduce that for any stopping time $\tau \in \mathcal{T}_{0,T}$,

$$E\, e^{(1+\epsilon)\left(\lambda_2 \int_0^\tau V_u^{v,a} du + a\varphi_{\lambda_1,\lambda_2}(T-\tau) + V_\tau^{v,a}\psi_{\lambda_1,\lambda_2}(T-\tau)\right)} \leq e^{a_\epsilon \varphi_{\lambda_1^\epsilon,\lambda_2^\epsilon}(T) + v\psi_{\lambda_1^\epsilon,\lambda_2^\epsilon}(T)},$$

where $a_\epsilon = (1+\epsilon)a$ et $\lambda_2^\epsilon = (1+\epsilon)\lambda_2$. Thus

$$\sup_{\tau \in \mathcal{T}_{0,T}} E\, e^{(1+\epsilon)\left(\lambda_2 \int_0^\tau V_u^{v,a} du + a\varphi_{\lambda_1,\lambda_2}(T-\tau) + V_\tau^{v,a}\psi_{\lambda_1,\lambda_2}(T-\tau)\right)} < +\infty \tag{A.2}$$

In conclusion, the local martingale $\left(M_s := e^{\lambda_2 \int_0^s V_u^{v,a} du} F^a_{\lambda_1,\lambda_2}(T-s, V_s^{v,a})\right)_{s \leq T}$ is uniformly integrable; it is a true martingale. In particular, we have

$$E\, e^{\lambda_2 \int_0^t V_u^{v,a} du + a\varphi_{\lambda_1,\lambda_2}(T-t) + V_t^{v,a}\psi_{\lambda_1,\lambda_2}(T-t)} = e^{a\varphi_{\lambda_1,\lambda_2}(T) + V_t^{v,a}\psi_{\lambda_1,\lambda_2}(T)}, \quad \forall t \leq T.$$

Thus

$$E\, e^{\lambda_2 \int_0^T V_u^{v,a} du + + \lambda_1 V_T^{v,a}} = e^{a\varphi_{\lambda_1,\lambda_2}(T) + V_t^{v,a}\psi_{\lambda_1,\lambda_2}(T)} \quad \square$$

B Solution of (2.1)

To solve (2.1), we write

$$\psi^2_{\lambda_1,\lambda_2}(t) - 2\frac{b}{\sigma^2}\psi_{\lambda_1,\lambda_2}(t) + 2\frac{\lambda_2}{\sigma^2} = \left(\psi_{\lambda_1,\lambda_2}(t) - \frac{b}{\sigma^2}\right)^2 + \frac{2\lambda_2\sigma^2 - b^2}{\sigma^4},$$

so there are three situations :

Case $2\lambda_2\sigma^2 < b^2$ **:** By setting

$$\bar{\psi}_{\lambda_1,\lambda_2}(t) = \psi_{\lambda_1,\lambda_2}(t) - \frac{b}{\sigma^2} \quad \text{and} \quad \alpha = \sqrt{b^2 - 2\lambda_2\sigma^2}/\sigma^2,$$

the equation (2.1) becomes

$$\bar{\psi}'_{\lambda_1,\lambda_2}(t) = \frac{\sigma^2}{2}\left(\bar{\psi}^2_{\lambda_1,\lambda_2}(t) - \alpha^2\right),$$

which gives

$$\frac{1}{2\alpha}\left(\frac{1}{\bar{\psi}_{\lambda_1,\lambda_2}(t) - \alpha} - \frac{1}{\bar{\psi}_{\lambda_1,\lambda_2}(t) + \alpha}\right)\bar{\psi}'_{\lambda_1,\lambda_2}(t) = \frac{\sigma^2}{2}.$$

Then,

$$\ln\left|\frac{\bar{\psi}_{\lambda_1,\lambda_2}(t) - \alpha}{\bar{\psi}_{\lambda_1,\lambda_2}(t) + \alpha}\right| = \alpha\sigma^2 t + \ln\left|\frac{\lambda_1 - \frac{b}{\sigma^2} - \alpha}{\lambda_1 - \frac{b}{\sigma^2} + \alpha}\right|$$

It follows that,

$$\begin{cases} \bar{\psi}_{\lambda_1,\lambda_2}(t) = -\alpha\frac{(Ce^{\alpha\sigma^2 t}+1)^2}{C^2e^{2\alpha\sigma^2 t}-1}, & \text{si } |\lambda_1 - \frac{b}{\sigma^2}| > \alpha, \\ \bar{\psi}_{\lambda_1,\lambda_2}(t) = -\alpha\frac{(Ce^{\alpha\sigma^2 t}-1)^2}{C^2e^{2\alpha\sigma^2 t}-1}, & \text{si } |\lambda_1 - \frac{b}{\sigma^2}| \leq \alpha, \end{cases}$$

where $C = \left|\frac{\lambda_1 - \frac{b}{\sigma^2} - \alpha}{\lambda_1 - \frac{b}{\sigma^2} + \alpha}\right|$. Thus

$$\begin{cases} \psi_{\lambda_1,\lambda_2}(t) = \frac{b}{\sigma^2} - \alpha\frac{Ce^{\alpha\sigma^2 t}+1}{Ce^{\alpha\sigma^2 t}-1}, & \text{if } |\lambda_1 - \frac{b}{\sigma^2}| > \alpha, \\ \psi_{\lambda_1,\lambda_2}(t) = \frac{b}{\sigma^2} - \alpha\frac{Ce^{\alpha\sigma^2 t}-1}{Ce^{\alpha\sigma^2 t}+1}, & \text{if } |\lambda_1 - \frac{b}{\sigma^2}| \leq \alpha. \end{cases}$$

Case $2\lambda_2\sigma^2 = b^2$ **:** The equation (2.1) can be written as

$$\bar{\psi}'_{\lambda_1,\lambda_2}(t) = \frac{\sigma^2}{2}\bar{\psi}^2_{\lambda_1,\lambda_2}(t),$$

which gives

$$\frac{1}{\bar{\psi}_{\lambda_1,\lambda_2}(0)} - \frac{1}{\bar{\psi}_{\lambda_1,\lambda_2}(t)} \quad = \quad \frac{\sigma^2 t}{2}.$$

Thus

$$\psi_{\lambda_1,\lambda_2}(t) = \frac{\lambda_1 - \frac{b}{\sigma^2}}{1 - \frac{\sigma^2}{2}(\lambda_1 - \frac{b}{\sigma^2})t} + \frac{b}{\sigma^2}.$$

Case $2\lambda_2\sigma^2 > b^2$ **:** We set $\beta = \sqrt{2\lambda_2\sigma^2 - b^2}/\sigma^2$. The equation (2.1) can be written as

$$\bar{\psi}'_{\lambda_1,\lambda_2}(t) = \frac{\sigma^2}{2}\left(\bar{\psi}^2_{\lambda_1,\lambda_2}(t) + \beta^2\right).$$

So

$$\frac{\bar{\psi}'_{\lambda_1,\lambda_2}(t)}{\bar{\psi}^2_{\lambda_1,\lambda_2}(t) + \beta^2} = \frac{\sigma^2}{2}$$

Integrating both sides of this equation we obtain

$$\frac{1}{\beta}\arctan(\bar{\psi}_{\lambda_1,\lambda_2}(t)/\beta) = \frac{\sigma^2 t}{2} + Cte.$$

Thus

$$\psi_{\lambda_1,\lambda_2}(t) = \frac{b}{\sigma^2} + \beta\tan\left(\beta\frac{\sigma^2 t}{2} + \arctan\left(\frac{\lambda_1\sigma^2 - b}{\beta\sigma^2}\right)\right).$$

Finally, the solution of (2.1) is given by the following table

$\lambda_2 < \frac{b^2}{2\sigma^2}$	$\begin{cases} \psi_{\lambda_1,\lambda_2}(t) = \frac{b}{\sigma^2} - \alpha\frac{Ce^{\alpha\sigma^2 t}+1}{Ce^{\alpha\sigma^2 t}-1}, & \text{if} \quad \lvert\lambda_1 - \frac{b}{\sigma^2}\rvert > \alpha, \\ \psi_{\lambda_1,\lambda_2}(t) = \frac{b}{\sigma^2} - \alpha\frac{Ce^{\alpha\sigma^2 t}-1}{Ce^{\alpha\sigma^2 t}+1}, & \text{if} \quad \lvert\lambda_1 - \frac{b}{\sigma^2}\rvert \leq \alpha, \end{cases}$ where $\alpha = \sqrt{b^2 - 2\lambda_2\sigma^2}/\sigma^2$ and $C = \left\lvert\frac{\lambda_1 - \frac{b}{\sigma^2} - \alpha}{\lambda_1 - \frac{b}{\sigma^2} + \alpha}\right\rvert$.
$\lambda_2 = \frac{b^2}{2\sigma^2}$	$\psi_{\lambda_1,\lambda_2}(t) = \frac{\lambda_1 - \frac{b}{\sigma^2}}{1 - \frac{\sigma^2}{2}(\lambda_1 - \frac{b}{\sigma^2})t} + \frac{b}{\sigma^2}.$
$\lambda_2 > \frac{b^2}{2\sigma^2}$	$\psi_{\lambda_1,\lambda_2}(t) = \frac{b}{\sigma^2} + \beta\tan\left(\beta\frac{\sigma^2 t}{2} + \arctan\left(\frac{\lambda_1\sigma^2 - b}{\beta\sigma^2}\right)\right),$ where $\beta = \sqrt{2\lambda_2\sigma^2 - b^2}/\sigma^2$.

and $t^*_{\lambda_1,\lambda_2}$ is given explicitly in terms of λ_1 and λ_2 by

$\lambda_2 < \frac{b^2}{2\sigma^2}$ and $\lambda_1 > \frac{b}{\sigma^2} + \sqrt{b^2 - 2\lambda_2\sigma^2}/\sigma^2$	$t^*_{\lambda_1,\lambda_2} = \frac{1}{\sqrt{b^2-2\lambda_2\sigma^2}} \ln\left(\frac{\lambda_1 - b/\sigma^2 + \sqrt{b^2-2\lambda_2\sigma^2}/\sigma^2}{\lambda_1 - b/\sigma^2 - \sqrt{b^2-2\lambda_2\sigma^2}/\sigma^2}\right).$
$\lambda_2 < \frac{b^2}{2\sigma^2}$ and $\lambda_1 \leq \frac{b}{\sigma^2} + \sqrt{b^2 - 2\lambda_2\sigma^2}/\sigma^2$	$t^*_{\lambda_1,\lambda_2} = +\infty.$
$\lambda_2 = \frac{b^2}{2\sigma^2}$	$t^*_{\lambda_1,\lambda_2} = \frac{2}{\sigma^2(\lambda_1 - \frac{b}{\sigma^2})}.$
$\lambda_2 > \frac{b^2}{2\sigma^2}$	$t^*_{\lambda_1,\lambda_2} = \frac{2}{\sqrt{2\lambda_2\sigma^2-b^2}}\left(\frac{\pi}{2} - \arctan\left(\frac{\lambda_1\sigma^2 - b}{\sqrt{2\lambda_2\sigma^2-b^2}}\right)\right).$

C Proof of Proposition 2.2

$\mu^*_+(t)$ is defined as solution of

$$t^*_{\lambda_1,\lambda_2}(\mu^*_+(t)) = t$$

We distinguish 2 cases

Case $t^*_{\lambda_1,\lambda_2}(\mu^*_+) = \frac{1}{\sqrt{b^2-2\lambda_2\mu^*_+\sigma^2}} \ln\left(\frac{\lambda_1\mu^*_+ - b/\sigma^2 + \sqrt{b^2-2\lambda_2\mu^*_+\sigma^2}/\sigma^2}{\lambda_1\mu^*_+ - b/\sigma^2 - \sqrt{b^2-2\lambda_2\mu^*_+\sigma^2}/\sigma^2}\right)$ **:** In this case we have, for $0 < \epsilon \ll 1$, $\psi_{\lambda_1(\mu^*_+-\epsilon),\lambda_2(\mu^*_+-\epsilon)}(t) = \frac{b}{\sigma^2} - \alpha\frac{Ce^{\alpha\sigma^2 t}+1}{Ce^{\alpha\sigma^2 t}-1}$., where $\alpha = \alpha(\epsilon) = \sqrt{b^2 - 2\lambda_2(\mu^*_+ - \epsilon)\sigma^2}/\sigma^2$ and $C = C(\epsilon) = \frac{\lambda_1(\mu^*_+-\epsilon) - \frac{b}{\sigma^2} - \alpha}{\lambda_1(\mu^*_+-\epsilon) - \frac{b}{\sigma^2} + \alpha}$. We obtain, by writing the Taylor expansion of $Ce^{\alpha\sigma^2 t}$ with respect to ϵ, that $\psi_{\lambda_1(\mu^*_+-\epsilon),\lambda_2(\mu^*_+-\epsilon)}(t)$ can be written as

$$\psi_{\lambda_1(\mu^*_+-\epsilon),\lambda_2(\mu^*_+-\epsilon)}(t) = \frac{b}{\sigma^2} + \alpha(0)\,\frac{2 - c_1\epsilon + \mathcal{O}(\epsilon^2)}{c_1\epsilon + \mathcal{O}(\epsilon^2)} = \frac{2\alpha(0)/c_1}{\epsilon} + e_1(\epsilon),$$

where $e_1(\epsilon)$ is bounded. It follows that, for ϵ small enough,

$$\ln\left(\mathbb{E}\, e^{(\mu^*_+-\epsilon)\lambda_1 V_t + (\mu^*_+-\epsilon)\lambda_2 \int_0^t V_u du}\right) - \frac{2v\alpha(0)/c_1}{\epsilon} \;\sim\; a\varphi_{\lambda_1(\mu^*_+-\epsilon),\lambda_2(\mu^*_+-\epsilon)}(t).$$

On the other hand, we have

$$\varphi_{\lambda_1(\mu^*_+-\epsilon),\lambda_2(\mu^*_+-\epsilon)}(t) = \frac{b}{\sigma^2}t - \alpha(\epsilon)\left(\ln\left(Ce^{\frac{\alpha\sigma^2}{2}t} - e^{-\frac{\alpha\sigma^2}{2}t}\right) - \ln(C-1)\right) \tag{C.1}$$

We obtain easily that $\varphi_{\lambda_1(\mu^*_+-\epsilon),\lambda_2(\mu^*_+-\epsilon)}(t) = \frac{2}{\sigma^2}\ln\frac{1}{\epsilon} + e_2(\epsilon)$, where $e_2(\epsilon)$ is bounded

Case $t^*_{\lambda_1,\lambda_2} = \frac{2}{\sqrt{2\lambda_2\sigma^2 - b^2}}\left(\frac{\pi}{2} - \arctan\left(\frac{\lambda_1\sigma^2 - b}{\sqrt{2\lambda_2\sigma^2-b^2}}\right)\right)$. **:** In this case, for $0 < \epsilon \ll 1$, we have

$$\begin{cases} \psi_{\lambda_1(\mu^*_+-\epsilon),\lambda_2(\mu^*_+-\epsilon)}(t) = \frac{b}{\sigma^2} + \frac{\sqrt{2\lambda_2(\mu^*_+-\epsilon)\sigma^2-b^2}}{\sigma^2}\tan\left(g(t,(\mu^*_+-\epsilon))\right), \\ \\ \varphi_{\lambda_1(\mu^*_+-\epsilon),\lambda_2(\mu^*_+-\epsilon)}(t) = \frac{b}{\sigma^2}t + \frac{2}{\sigma^2}\left(\ln\cos g(0,\mu^*_+-\epsilon) - \ln\cos g(t,\mu^*_+-\epsilon)\right), \end{cases}$$

where

$$g(t,\mu) = \frac{\sqrt{2\mu\lambda_2\sigma^2 - b^2}}{2}t + \arctan\left(\frac{\lambda_1\mu\sigma^2 - b}{\sqrt{2\mu\lambda_2\sigma^2-b^2}}\right) \tag{C.2}$$

We get by the same as before that for ϵ small enough,

$$\ln\left(\mathbb{E}\, e^{(\mu^*_+-\epsilon)\lambda_1 V_t + (\mu^*_+-\epsilon)\lambda_2\int_0^t V_u du}\right) = \frac{x\sqrt{2\mu^*_+\sigma^2 - b^2}}{\sigma^2\partial_\mu g(t,\mu^*_+)}\frac{1}{\epsilon} - \frac{2a}{\sigma^2}\ln(\epsilon) + e(\epsilon).$$

where $e(\epsilon)$ is bounded.

D Proof of Theorem 3.2

D.1 Upper Bound

A simple application of Markov's inequality shows that, for every $\mu > 0$,

$$\mathbb{P}(Z > R) = \mathbb{P}(e^{\mu Z} > e^{\mu R}) \le e^{-\mu R}\,\mathbb{E}[e^{\mu Z}] = m\, e^{\mu + \Lambda(\mu)}.$$

In particular, for $\mu = \mu^* - \sqrt{\frac{\omega}{R}}$, we have

$$\ln\mathbb{P}(Z > R) \le -(\mu^* - \sqrt{\frac{\omega}{R}})R + \Lambda(\mu^* - \sqrt{\frac{\omega}{R}}) = -\mu^* R + 2\sqrt{\omega R} + \nu\ln\sqrt{\frac{R}{\omega}}$$

It follows that,

$$\limsup_{R\to\infty}\frac{\ln\mathbb{P}(Z > R) + \mu^* R - 2\sqrt{\omega R}}{\ln R} \le \frac{\nu}{2} \tag{D.1}$$

D.2 Lower Bound for (3.1)

In order to derive a lower bound for the lim inf for (3.1) we proceed as follows. Given any $p > 0$ and any $R > 0$, we have

$$\begin{aligned}
\mathbb{E}\left(e^{pZ}\right) &= \mathbb{E}\left(e^{pZ}\mathbb{1}_{Z\leq R}\right) + \mathbb{E}\left(e^{pZ}\mathbb{1}_{Z>R}\right) \\
&= \mathbb{E}\left(\int_0^Z pe^{pr}dr\mathbb{1}_{Z\leq R}\right) + \mathbb{E}\left(e^{pZ}\mathbb{1}_{Z>R}\right) \\
&\leq 1 + \mathbb{E}\left(\int_0^Z pe^{pr}\mathbb{1}_{0<r\leq Z\leq R}dr\right) + \mathbb{E}\left(e^{pZ}\mathbb{1}_{Z>R}\right) \\
&\leq 1 + \int_0^R pe^{pr}\mathbb{P}(Z>r)dr + \mathbb{E}\left(e^{pZ}\mathbb{1}_{Z>R}\right) \qquad \text{(D.2)}
\end{aligned}$$

We choose $p = \mu^* - \sqrt{\omega}R^{-\beta}$, where $\beta \in]\frac{1}{2}, 1[$ to be given bellow. Denote by $J := \int_0^R e^{pr}\mathbb{P}(Z_t > r)dr$. Using (D.1) we have

$$\begin{aligned}
J &\leq m\int_0^R e^{-\sqrt{\omega}R^{-\beta}+2\sqrt{\omega z}+\nu z}dz = mR^{\nu+1}e^{\sqrt{\omega}R^{\beta}}\int_0^1 z^{\nu}e^{-\sqrt{\omega}R^{1-\beta}(\sqrt{z}-R^{\beta-\frac{1}{2}})^2}dz \\
&\leq mR^{\nu+1}e^{\sqrt{\omega}R^{\beta}}e^{-\sqrt{\omega}R^{1-\beta}(1-R^{\beta-\frac{1}{2}})^2} = me^{\Lambda(p)}R^{\nu/2+1}e^{-\sqrt{\omega}R^{1-\beta}(1-R^{\beta-\frac{1}{2}})^2}
\end{aligned}$$

Let's set

$$\beta = \frac{1}{2} + \left(\frac{\eta}{\sqrt{\omega R}\,\ln R}\right)^{\frac{1}{2}} \qquad \text{(D.3)}$$

For some positive η. We can then write R^{β} as

$$R^{\beta} = R^{\frac{1}{2}}\ e^{(\beta-\frac{1}{2})\ln R} \sim R^{\frac{1}{2}}\ (1 + (\beta - \frac{1}{2})\ln R) = R^{\frac{1}{2}}\ (1 + \left(\frac{\eta\ \ln R}{\sqrt{\omega R}}\right)^{\frac{1}{2}})$$

It follows that $(1 - R^{\beta-\frac{1}{2}})^2$ is given as

$$(1 - R^{\beta-\frac{1}{2}})^2 \sim \frac{\eta\ \ln R}{\sqrt{\omega R}}$$

Thus

$$J \leq me^{\Lambda(p)}\ R^{\frac{\nu}{2}+1-\eta}$$

We finally get, by taking $\eta = \frac{\nu}{2} + 4$

$$\mathbb{E}\left(e^{pZ}\mathbb{1}_{Z>R}\right) \geq e^{\Lambda(p)}\left(1 - R^{-3}\right) \qquad \text{(D.4)}$$

Applying Hölder's inequality to $\mathbb{E}\left(e^{pZ}\mathbb{1}_{Z>R}\right)$, for some $q>1$, we get

$$\mathbb{E}\left(e^{pZ}\mathbb{1}_{Z>R}\right) \leq \left(\mathbb{E}\ e^{qpZ}\right)^{\frac{1}{q}} \left(\mathbb{P}(Z>R)\right)^{1-\frac{1}{q}}.$$

It follows that for any $q>1$, we have

$$(1-\frac{1}{q})\ln\mathbb{P}(Z>R) \geq \ln\mathbb{E}\left(e^{pZ}\mathbb{1}_{Z>R}\right) - \frac{1}{q}\Lambda(qp)$$

Thus

$$\ln\mathbb{P}(Z>R) \quad \geq \quad \frac{q}{q-1}\left(\Lambda(p) - \frac{1}{q}\Lambda(qp) + \ln\left(1-R^{-3}\right)\right) \tag{D.5}$$

We set

$$q(R) = \frac{\Lambda^{-1}(\frac{\Lambda(p)}{\delta_q})}{p}, \quad \text{with} \quad \delta_q = \frac{1}{R}. \tag{D.6}$$

In particular, we have

$$\Lambda(q(R)p) \quad = \quad \frac{\Lambda(p)}{\delta_q}$$

Thus

$$\begin{aligned} \frac{q(R)}{q(R)-1}(\Lambda(p) - \frac{1}{q}\Lambda(qp)) \quad &= \quad \frac{q(R)}{q(R)-1}\left(1-\frac{1}{q\delta_q}\right)\Lambda(p) \\ &= \quad \frac{q\delta_q - 1}{\delta_q(q-1)}\Lambda(p). \end{aligned}$$

On the other hand, for R large enough, we have

$$\Lambda^{-1}(\frac{\Lambda(p)}{\delta_q}) = \mu^* - \delta_q\sqrt{\omega}R^{-\beta} - \delta(\delta-1)\nu\frac{\ln R^\beta}{R^{2\beta}}$$

It follows that

$$\begin{aligned} \frac{q\delta_q - 1}{q-1} \quad &= \quad \frac{-\mu^*(1-\delta_q) + (1-\delta_q^2)\sqrt{\omega}R^{-\beta} - \delta^2(\delta-1)\nu\frac{\ln R^\beta}{R^{2\beta}}}{(1-\delta_q)\sqrt{\omega}R^{-\beta} - \delta(\delta-1)\nu\frac{\ln R^\beta}{R^{2\beta}}} \\ &= \quad \frac{-\mu^* + (1+\delta_q)\sqrt{\omega}R^{-\beta} + \delta^2\nu\frac{\ln R^\beta}{R^{2\beta}}}{\sqrt{\omega}R^{-\beta} + \delta\nu\frac{\ln R^\beta}{R^{2\beta}}} \end{aligned}$$

Thus
$$\frac{q\delta_q-1}{\delta_q(q-1)}\Lambda(p)=\left(-\mu^*+(1+\delta_q)\sqrt{\omega}R^{-\beta}+\delta^2\nu\frac{\ln R^\beta}{R^{2\beta}}\right)\frac{\sqrt{\omega}R^\beta+\nu\ln R^\beta}{\sqrt{\omega}R^{-\beta}+\delta\nu\frac{\ln R^\beta}{R^{2\beta}}}\frac{1}{\delta_q}$$
On the other hand, we have
$$\frac{1}{\delta_q}\frac{\sqrt{\omega}R^\beta+\nu\ln R^\beta}{\sqrt{\omega}R^{-\beta}+\delta\nu\frac{\ln R^\beta}{R^{2\beta}}} \quad = \quad R^{2\beta}\frac{1+\nu\frac{\ln R^\beta}{R^\beta}}{1+\delta\nu\frac{\ln R^\beta}{R^\beta}}=\frac{R^{2\beta}}{\delta_q}(1-\frac{1}{R})+\mathcal{O}(\frac{1}{R^2})$$
On the other hand, as $\frac{q}{q-1}\sim R^{-1-\beta}$ we have $\frac{q}{q-1}\ln(1-R^{-3})=\mathcal{O}(R)$. We then get
$$\ln\mathbb{P}(Z>R)\geq\left(-\mu^*+(1+\delta_q)\sqrt{\omega}R^{-\beta}+\delta^2\nu\frac{\ln R^\beta}{R^{2\beta}}\right)\left(R^{2\beta}(1-\frac{1}{R})\right)-\mathcal{O}(R^{-1})$$
Writing
$$R^{2\beta}=R\ R^{2\beta-1}=R\ e^{(2\beta-1)\ln R}\sim R\left(1+2\left(\frac{\eta\ \ln R}{\sqrt{\omega R}}\right)^{\frac{1}{2}}\right)$$
We conclude that for any $\alpha>\frac{3}{4}$, we have
$$\liminf_{R\to+\infty}\frac{\ln\mathbb{P}(Z>R)+\mu^*R}{R^\alpha}\geq 0. \tag{D.7}$$

D.3 Lower Bound for (3.2)

In order to derive a lower bound for the lim sup for (3.2) we proceed as follows. Assume that
$$\limsup_{R\to\infty}\frac{\ln\mathbb{P}(Z>R)+\mu^*R-2\sqrt{\omega R}}{\ln(R)}=\nu^*<\frac{\nu}{2}-\frac{3}{4}$$
we will show that this will lead to a contradiction.

We first write for $p=\mu^*-\sqrt{\frac{\omega}{R}}$,
$$\begin{aligned} e^{\Lambda(p)}=\mathbb{E}\left(e^{pZ}\right) &= \mathbb{E}\left(e^{pZ}\mathbb{1}_{Z\leq R^{\frac{1}{4}}}\right)+\mathbb{E}\left(e^{pZ}\mathbb{1}_{Z>R^{\frac{1}{4}}}\right)\\ &\leq e^{pR^{\frac{1}{4}}}+m\int_{R^{\frac{1}{4}}}^{\infty}e^{-\sqrt{\frac{\omega}{R}}z+2\sqrt{\omega z}+\nu'\ln z}dz\\ &= e^{pR^{\frac{1}{4}}}+me^{\sqrt{\omega R}}\ R^{\nu'+1}\int_{R^{-\frac{3}{4}}}^{\infty}z^{\nu'}\ e^{-\sqrt{\omega R}(\sqrt{z}-1)^2}dz \end{aligned}$$

where $\nu'\in]\nu^*,\frac{\nu}{2}-1[$.

Lemma D.1. *For any $\gamma \in \mathbb{R}$, there exist $c > 0$ so that*

$$\int_{R^{-\frac{3}{4}}}^{\infty} z^{\gamma}\, e^{-\sqrt{\omega R}(\sqrt{z}-1)^2} dz = cR^{-\frac{1}{4}} + \mathcal{O}(R^{-\frac{1}{2}}) \tag{D.8}$$

It follows that

$$M\, e^{\sqrt{\omega R}}\, R^{\frac{\nu}{2}} = e^{\Lambda(p)} \quad \leq \quad e^{pR^{\frac{1}{4}}} + 2c\, m\, e^{\sqrt{\omega R}}\, R^{\nu' + \frac{3}{4}}$$

which is impossible as $\nu' + \frac{3}{4} < \frac{\nu}{2}$.

Thus

$$\limsup_{R\to+\infty} \frac{\ln \mathbb{P}(Z > R) + \mu^* R - 2\sqrt{\omega R}}{\ln(R)} \geq \frac{\nu}{2} - \frac{3}{4}. \tag{D.9}$$

D.4 Proof of (3.3)

Suppose

$$\lim_{R\to\infty} \frac{\ln \mathbb{P}(Z > R) + \mu^* R - 2\sqrt{\omega R}}{\ln(R)} = \nu^*$$

we will show that $\nu^* = \frac{\nu}{2} - \frac{3}{4}$. For this we will show that both $\left\{\nu^* < \frac{\nu}{2} - \frac{3}{4}\right\}$ and $\left\{\nu^* > \frac{\nu}{2} - \frac{3}{4}\right\}$ lead to a contradiction.

Assume $\nu^* < \frac{\nu}{2} - \frac{3}{4}$. For $p = \mu^* - \sqrt{\frac{\omega}{R}}$, we have

$$\begin{aligned} M_1\, e^{\sqrt{\omega R} + \frac{\nu}{2}\ln(R)} = e^{\Lambda(p)} \quad &\leq \quad e^{pR^{\frac{1}{4}}} + M_2 \int_{R^{\frac{1}{4}}}^{\infty} e^{-\sqrt{\frac{\omega}{R}} z + 2\sqrt{\omega z} + \nu' \ln z} dz \\ &= \quad e^{pR^{\frac{1}{4}}} + M_2\, e^{\sqrt{\omega R}}\, R^{\nu'+1} \int_{R^{-\frac{3}{4}}}^{\infty} z^{\nu'}\, e^{-\sqrt{\omega R}(\sqrt{z}-1)^2} dz \end{aligned}$$

where $\nu' \in]\nu^*, \frac{\nu}{2} - \frac{3}{4}[$. Using Lemma D.1, this means that

$$M_1\, e^{\sqrt{\omega R}}\, R^{\frac{\nu}{2}} \quad \leq \quad e^{pR^{\frac{1}{4}}} + c\, M_2\, e^{\sqrt{\omega R}}\, R^{\nu' + \frac{3}{4}} \sim c\, M_2\, e^{\sqrt{\omega R}}\, R^{\frac{\nu}{2}}\, R^{-(\frac{\nu}{2} - \frac{3}{4} - \nu')}$$

which is a contradiction.

On the other hand, if we assume $\nu^* > \frac{\nu}{2} - \frac{3}{4}$, we will get

$$M_1\, e^{\sqrt{\omega R}}\, R^{\frac{\nu}{2}} \quad \geq \quad e^{pR^{\frac{1}{4}}} + c_2\, M_2\, e^{\sqrt{\omega R}}\, R^{\nu' + \frac{3}{4}} \sim c_2\, M_2\, e^{\sqrt{\omega R}}\, R^{\frac{\nu}{2}}\, R^{\nu'' - (\frac{\nu}{2} - \frac{3}{4})}$$

with $\nu'' > \in]\nu^*, \frac{\nu}{2} - \frac{3}{4}[$ $\square$.

Proof of Lemma D.1

We note first that $\int_{R^{-\frac{3}{4}}}^{\infty} z^{\gamma}\, e^{-\sqrt{\omega R}(\sqrt{z}-1)^2} dz = 2\int_{R^{-\frac{3}{8}}}^{\infty} z^{2\gamma+1}\, e^{-\sqrt{\omega R}(z-1)^2} dz$. Denote by $I(\gamma) := \int_{R^{-\frac{3}{4}}}^{1} z^{\gamma}\, e^{-\sqrt{\omega R}(\sqrt{z}-1)^2} dz$ and $J(\gamma) := \int_{1}^{\infty} z^{\gamma}\, e^{-\sqrt{\omega R}(\sqrt{z}-1)^2} dz$. One can easily see that if $\gamma > 0$

$$I([\gamma]) \geq I(\gamma) > I([\gamma]+1) \quad \text{and} \quad J([\gamma]) \leq J(\gamma) < J([\gamma]+1)$$

and if $\gamma < 0$

$$I([\gamma]) \leq I(\gamma) < I([\gamma]+1) \quad \text{and} \quad J([\gamma]) \geq J(\gamma) > J([\gamma]+1)$$

which means that for any $\gamma \in \mathbb{R}$ there exist $n_1, n_2, n_3, n_4 \in \mathbb{Z}$ so that

$$I(n_1) + J(n_2) \leq I(\gamma) + J(\gamma) \leq I(n_3) + J(n_3) \tag{D.10}$$

Now for any $n \in \mathbb{Z}$, we have

$$\begin{aligned} I(n) - I(n-1) &= \int_{R^{-\frac{3}{8}}}^{1} z^{n-1}(z-1)\, e^{-\sqrt{\omega R}(z-1)^2} dz \\ &= \frac{R^{-\frac{3(n-1)}{8}}\; e^{-\sqrt{\omega R}(R^{-\frac{3}{8}}-1)^2} - e^{-\sqrt{\omega R}}}{\sqrt{\omega R}} + \frac{n-1}{\sqrt{\omega R}} I(n-2) \end{aligned}$$

Using this equality we can easily check that

$$I(n) = I(0) + \mathcal{O}(R^{-\frac{1}{2}})$$

By the same way, we get

$$J(n) = J(0) + \mathcal{O}(R^{-\frac{1}{2}})$$

where

$$I(0) = \int_{R^{-\frac{3}{8}}}^{1} e^{-\sqrt{\omega R}(z-1)^2} dz = \int_{0}^{1-R^{-\frac{3}{8}}} e^{-\sqrt{\omega R} z^2} dz$$

To calculate $I(0)$ we consider the function $I_0(R) := \int_0^{1-R^{-\frac{3}{8}}} e^{-\sqrt{\omega R} z^2} dz$. This function is the unique solution of

$$I_0'(R) = \frac{(1+\frac{1}{2}R^{-\frac{3}{8}})\; e^{-(1-R^{-\frac{3}{8}})^2\sqrt{\omega R}}}{4R} - \frac{1}{4R} I_0(R), \quad f(1) = 0.$$

The solution of this equation is given as

$$\begin{aligned} I_0(R) &= R^{-\frac{1}{4}} \int_1^R \frac{(1+\frac{1}{2}y^{-\frac{3}{8}})\ e^{-(1-y^{-\frac{3}{8}})^2\sqrt{\omega y}}}{4y^{\frac{3}{4}}} dy \\ &= R^{-\frac{1}{4}} \left[\int_1^\infty \frac{(1+\frac{1}{2}y^{-\frac{3}{8}})\ e^{-(1-y^{-\frac{3}{8}})^2\sqrt{\omega y}}}{4y^{\frac{3}{4}}} dy - \int_R^\infty \frac{(1+\frac{1}{2}y^{-\frac{3}{8}})\ e^{-(1-y^{-\frac{3}{8}})^2\sqrt{\omega y}}}{4y^{\frac{3}{4}}} dy \right] \\ &= c_0(I)\ R^{-\frac{1}{4}} + \mathcal{O}(R^{-1}) \end{aligned}$$

where

$$c_0(I) = \int_1^\infty \frac{(1+\frac{1}{2}y^{-\frac{3}{8}})\ e^{-(1-y^{-\frac{3}{8}})^2\sqrt{\omega y}}}{4y^{\frac{3}{4}}} dy \tag{D.11}$$

.

By the same way we get $J_0'(R) = \frac{1}{4R}J_0(R)$. Then

$$J_0(R) = c_0(J)\ R^{-\frac{1}{4}}$$

where

$$c_0(J) = J_0(1) = \int_0^\infty e^{-\sqrt{\omega}z^2} \tag{D.12}$$

We finally get for any $\gamma \in \mathbb{R}$

$$(c_0(I)+c_0(J))R^{-\frac{1}{4}} + \mathcal{O}_1(R^{-\frac{1}{2}}) \le I(\gamma)+J(\gamma) \le (c_0(I)+c_0(J))R^{-\frac{1}{4}} + \mathcal{O}_2(R^{-\frac{1}{2}}) \tag{D.13}$$

E Proof of Proposition 4.1

We have $F_p(t) = e^{a\varphi_p(t)+v\psi_p(t)}$, where $\varphi_p(t) = \int_0^t \psi_p(s)ds$ and ψ_p is given by theorem 2.1 as $F_p(t) = e^{a\varphi_p(t)+v\psi_p(t)}$, where $\varphi_p(t) = \int_0^t \psi_p(s)ds$ and ψ_p is given by

$\frac{p^2-p}{2} < \frac{(b-\rho\sigma p)^2}{2\sigma^2}$	$\begin{cases} \psi_p(t) = \frac{b-\rho\sigma p}{\sigma^2} - \alpha\frac{Ce^{\alpha\sigma^2 t}+1}{Ce^{\alpha\sigma^2 t}-1}, & \text{if } \lvert\frac{b-\rho\sigma p}{\sigma^2}\rvert > \alpha, \\ \psi_p(t) = \frac{b-\rho\sigma p}{\sigma^2} - \alpha\frac{Ce^{\alpha\sigma^2 t}-1}{Ce^{\alpha\sigma^2 t}+1}, & \text{if } \lvert\frac{b-\rho\sigma p}{\sigma^2}\rvert \le \alpha, \end{cases}$ where $\alpha = \sqrt{(b-\rho\sigma p)^2 - p(p-1)\sigma^2}/\sigma^2$ and $C = \left\lvert\frac{b-\rho\sigma p+\alpha\sigma^2}{b-\rho\sigma p-\alpha\sigma^2}\right\rvert$.
$\frac{p^2-p}{2} = \frac{(b-\rho\sigma p)^2}{2\sigma^2}$	$\psi_p(t) = \frac{(b-\rho\sigma p)^2}{2\sigma^2(1+\frac{b-\rho\sigma p}{2}t)}$.
$\frac{p^2-p}{2} > \frac{(b-\rho\sigma p)^2}{2\sigma^2}$	$\psi_p(t) = \frac{b-\rho\sigma p}{\sigma^2} + \beta\tan\left(\beta\frac{\sigma^2 t}{2} + \arctan\left(\frac{-b+\rho\sigma p}{\beta\sigma^2}\right)\right)$, where $\beta = \sqrt{2\frac{p^2-p}{2}\sigma^2 - (b-\rho\sigma p)^2/\sigma^2}$.

where ψ_p is defined for $t \in [0, t_p^*[$, where t_p^* is given by

$0 < p^2 - p < \frac{(b-\rho\sigma p)^2}{\sigma^2}$ and $b - \rho\sigma p < 0$	$t_p^* = \frac{1}{\sqrt{(b-\rho\sigma p)^2 - p(p-1)\sigma^2}} \ln\left(\frac{b-\rho\sigma p - \sqrt{(b-\rho\sigma p)^2 - p(p-1)\sigma^2}}{b-\rho\sigma p + \sqrt{(b-\rho\sigma p)^2 - p(p-1)\sigma^2}}\right).$
$p^2 - p < \frac{(b-\rho\sigma p)^2}{\sigma^2}$ and $b - \rho\sigma p \geq 0$	$t_p^* = +\infty.$
$p^2 - p = \frac{(b-\rho\sigma p)^2}{\sigma^2}$ and $b - \rho\sigma p < 0$	$t_{\lambda_1,\lambda_2}^* = \frac{-2}{b-\rho\sigma p}.$
$p^2 - p > \frac{(b-\rho\sigma p)^2}{\sigma^2}$	$t_p^* = \frac{2\left(\pi \mathbb{1}_{b-\rho\sigma p>0} - \arctan\left(\frac{\sqrt{p(p-1)\sigma^2 - (b-\rho\sigma p)^2}}{b-\rho\sigma p}\right)\right)}{\sqrt{p(p-1)\sigma^2 - (b-\rho\sigma p)^2}}.$

Note that if $p(p-1) < 0$, then $t_p^* = +\infty$.

Consider the critical moment of e^X. For $t > 0$, denote by

$$\mu_+^X(t) := \inf\left\{p > 0 : \quad t_p^* = t\right\} \quad \text{and} \quad \mu_-^X(t) := -\sup\left\{p < 0 : \quad t_p^* = t\right\} \tag{E.1}$$

Denote by

$$t_1(p) := \frac{1}{\sqrt{(b-\rho\sigma p)^2 - p(p-1)\sigma^2}} \ln\left(\frac{b-\rho\sigma p - \sqrt{(b-\rho\sigma p)^2 - p(p-1)\sigma^2}}{b-\rho\sigma p + \sqrt{(b-\rho\sigma p)^2 - p(p-1)\sigma^2}}\right) \tag{E.2}$$

and by

$$t_2(p) := \frac{2(\pi \mathbb{1}_{b-\rho\sigma p>0} - \arctan\left(\frac{\sqrt{p(p-1)\sigma^2 - (b-\rho\sigma p)^2}}{b-\rho\sigma p})\right)}{\sqrt{p(p-1)\sigma^2 - (b-\rho\sigma p)^2}}. \tag{E.3}$$

Also denote by $p_0^{\pm} := \frac{1 - \frac{2b\rho}{\sigma} \pm \sqrt{(1-\frac{2b\rho}{\sigma})^2 + 4(1-\rho^2)\frac{b^2}{\sigma^2}}}{2(1-\rho^2)}$ (the solutions of $p^2 - p = \frac{(b-\rho\sigma p)^2}{\sigma^2}$). Then we have

$$\mu_-^X = -\sup\left\{p < p_0^- : \ t_2(p) = t\right\} \tag{E.4}$$

and

$$\mu_+^X(t) = \begin{cases} \inf\{p > 1 : \ t_1(p) = t\}, & \text{If } \left\{\rho > 0, \ b - \rho\sigma < 0 \text{ and } \frac{2}{\rho\sigma p_0^+ - b} \leq t\right\}. \\ \inf\left\{p > p_0^+ : \ t_2(p) = t\right\}, & \text{Otherwise.} \end{cases} \tag{E.5}$$

Case $\rho > 0, \ b - \rho\sigma < 0$ **and** $\frac{2}{\rho\sigma p_0^+ - b} \leq t$ **:** In this case, $\mu_+^X(t)$ is solution of $t_1(\mu_+^X(t)) = t$. Which means that for ϵ small enough and for any $p \in [\mu_+^X(t) - \epsilon, \mu_+^X(t)[$, we have

$$\psi_p(t) = \frac{b-\rho\sigma p}{\sigma^2} - \alpha \frac{Ce^{\alpha\sigma^2 t} + 1}{Ce^{\alpha\sigma^2 t} - 1}$$

and

$$\varphi_p(t) = \frac{b-\rho\sigma p}{\sigma^2} t - \frac{2}{\sigma^2}\left(\ln\left(Ce^{\alpha\sigma^2 t/2} - e^{-\alpha\sigma^2 t/2}\right) - \ln(C-1)\right)$$

where

$$\alpha \equiv \alpha(p) = \sqrt{(b-\rho\sigma p)^2 - p(p-1)\sigma^2} \text{ and } C \equiv C(p) = \frac{-b+\rho\sigma p - \alpha\sigma^2}{-b+\rho\sigma p + \alpha\sigma^2} \tag{E.6}$$

We can easily check that $\psi_{\mu_+^X-\epsilon}(t)$ and $\varphi_{\mu_+^X-\epsilon}(t)$ can be writen as

$$\psi_{\mu_+^X-\epsilon}(t) = \frac{\omega_+}{v\epsilon} + \mathcal{O}(1) \text{ and } \varphi_{\mu_+^X-\epsilon}(t) = \frac{2}{\sigma^2}\ln(\frac{1}{\epsilon}) + \mathcal{O}(1)$$

where

$$\omega_+ = v\ \alpha(\mu_+^X)\frac{C(\mu_+^X)e^{\alpha(\mu_+^X)\sigma^2 t}+1}{\frac{d}{dp}[C(p)e^{\alpha(p)\sigma^2 t}]_{p=\mu_+^X}} \tag{E.7}$$

General case : We have

$$\mu_+^X(t) := \inf\{p>0: \quad t_2(p)=t\} \quad \text{and} \quad \mu_-^X(t) := -\sup\{p<0: \quad t_2(p)=t\}$$

where $t_2(p) := \frac{2(\pi\mathbb{1}_{b-\rho\sigma p>0} - \arctan\left(\frac{\sqrt{p(p-1)\sigma^2-(b-\rho\sigma p)^2}}{b-\rho\sigma p}\right))}{\sqrt{p(p-1)\sigma^2-(b-\rho\sigma p)^2}}$. In particular for any $p \in [\mu_+^X(t)-\epsilon, \mu_+^X(t)[$, we have

$$\psi_p(t) = \frac{b-\rho\sigma p}{\sigma^2} + \beta\tan\left(\beta\frac{\sigma^2 t}{2} + \arctan\left(\frac{-b+\rho\sigma p}{\beta\sigma^2}\right)\right)$$

and

$$\varphi_p(t) = \frac{b-\rho\sigma p}{\sigma^2}t + \frac{2}{\sigma^2}\ln\left(\frac{\arctan\left(\frac{-b+\rho\sigma p}{\beta\sigma^2}\right)}{\beta\frac{\sigma^2 t}{2} + \arctan\left(\frac{-b+\rho\sigma p}{\beta\sigma^2}\right)}\right)$$

where

$$\beta \equiv \beta(p) = \sqrt{2\frac{p^2-p}{2}\sigma^2 - (b-\rho\sigma p)^2/\sigma^2} \tag{E.8}$$

Denote by

$$g(t,p) := \beta\frac{\sigma^2 t}{2} + \arctan\left(\frac{-b+\rho\sigma p}{\beta\sigma^2}\right) \tag{E.9}$$

We can easily check that

$$\psi_{\mu_+^X-\epsilon} = \frac{\omega_+}{v\ \epsilon} + \mathcal{O}(1) \text{ and } \varphi_{\mu_+^X-\epsilon} = \frac{2}{\sigma^2}\ln(\frac{1}{\epsilon}) + \mathcal{O}(1)$$

where

$$\omega_+ := v\ \frac{\beta(\mu_+^X)}{\frac{d}{dp}[g(t,p)]_{p=\mu_+^X}} \tag{E.10}$$

By the same way, we show that for any $p \in]-\mu_-^X, -\mu_-^X + \epsilon]$, we have

$$\psi_p(t) = \frac{b - \rho\sigma p}{\sigma^2} + \beta(p)\tan\left(g(t,p)\right)$$

and

$$\varphi_p(t) = \frac{b - \rho\sigma p}{\sigma^2} t + \frac{2}{\sigma^2}\ln\left(\frac{\cos g(0,p)}{\cos g(t,p)}\right)$$

It follows that

$$\psi_{-\mu_+^X+\epsilon} = \frac{\omega_-}{v\ \epsilon} + \mathcal{O}(1) \text{ and } \varphi_{-\mu_+^X+\epsilon} = \frac{2}{\sigma^2}\ln(\frac{1}{\epsilon}) + \mathcal{O}(1)$$

where

$$\omega_- := \frac{-v\ \beta(-\mu_-^X)}{\frac{d}{dp}[g(t,p)]_{p=-\mu_-^X}} \tag{E.11}$$

Note that $\omega_\pm > 0$, for any $t > 0$.

F Proof of Theorem 4.2

This theorem is very similar to Theorem 3.1 in [4] where having a sharp asymptotic formulas for the Call price as $e^{-A_1 x + A_2\sqrt{x}} x^\nu$, an asymptotic formulas for the implied volatility is obtained as in the theorem. More precisely, it is shown in [4] (cf Lemma 10.1 of) that the solution σ of $e^{-A_1 x + A_2\sqrt{x}} x^\nu \approx_{k\to+\infty} N\left(\frac{-k+\frac{t}{2}\sigma_t^2(k)}{\sqrt{t}\sigma_t(k)}\right) - e^k N\left(\frac{-k-\frac{t}{2}\sigma_t^2(k)}{\sqrt{t}\sigma_t(k)}\right)$ is given as

$$\sigma_t(k) \approx \beta_1(A)\sqrt{k} + \beta_2(A_1)A_2 + \beta_2(A_1)(\nu + \frac{1}{2})\frac{\log k}{\sqrt{k}} \tag{F.1}$$

where $\beta_1(A) = \frac{\sqrt{2}}{\sqrt{t}}\left(\sqrt{A+1} - \sqrt{A}\right)$ and $\beta_2(A) = \frac{1}{\sqrt{2t}}\left(\frac{1}{\sqrt{A}} - \frac{1}{\sqrt{A+1}}\right)$.

Now suppose (3.3) holds for $\pm X_t$, then

$$\mathbb{P}(X_t > k) \approx_{k\to+\infty} e^{-\mu_+^X k + 2\sqrt{\omega^+ x}} x^{\frac{a}{\sigma^2} - \frac{3}{4}} \text{ and } \mathbb{P}(X_t < -k) \approx_{k\to+\infty} e^{-\mu_-^X|x| + 2\sqrt{\omega^+|x|}} |x|^{\frac{a}{\sigma^2} - \frac{3}{4}}$$

In particular, we have $\mathbb{E}\left(e^X - e^k\right)_+ \approx_{k\to+\infty} e^{-(\mu_+^X - 1)k + 2\sqrt{\omega_+^X k}} k^{\frac{a}{\sigma^2} - \frac{3}{4}}$ and $\mathbb{E}\left(e^{-k} - e^X\right)_+ \approx_{k\to+\infty} e^{-(1+\mu_-^X)k + 2\sqrt{\omega^+ k}} k^{\frac{a}{\sigma^2} - \frac{3}{4}}$. It follows that $\sigma_t(k)$ and $\sigma_t(-k)$, for k large enough, are given as solution to $e^{-(\mu_+^X - 1)k + 2\sqrt{\omega_+^X k}} k^{\frac{a}{\sigma^2} - \frac{3}{4}} \approx N\left(\frac{-k+\frac{t}{2}\sigma_t^2(k)}{\sqrt{t}\sigma_t(k)}\right) - e^k N\left(\frac{-k-\frac{t}{2}\sigma_t^2(k)}{\sqrt{t}\sigma_t(k)}\right)$ and $e^{-(\mu_-^X + 1)k + 2\sqrt{\omega_-^X k}} k^{\frac{a}{\sigma^2} - \frac{3}{4}} \approx e^{-k} N\left(\frac{-k+\frac{t}{2}\sigma_t^2(-k)}{\sqrt{t}\sigma_t(-k)}\right) - N\left(\frac{-k-\frac{t}{2}\sigma_t^2(-k)}{\sqrt{t}\sigma_t(-k)}\right)$. Thus

$$\sigma_t(k) \approx \beta_1(\mu_+^X - 1)\sqrt{k} + 2\beta_2(\mu_+^X - 1)\sqrt{\omega} + \beta_2(\mu_+^X - 1)(\frac{a}{2} - \frac{1}{4})\frac{\log k}{\sqrt{k}}$$

and

$$\sigma_t(-k) \approx \beta_1(\mu_-^X)\sqrt{k} + 2\beta_2(\mu_-^X)\sqrt{\omega} + \beta_2(\mu_-^X)(\frac{a}{2} - \frac{1}{4})\frac{\log k}{\sqrt{k}}$$

Now as the Call price is an increasing function of the implied volatility, an upper bound of the distribution given as $e^{\mu^* x + 2\sqrt{\omega x}} x^{\nu^*}$ will give an upper bound of the implied volatility. So we can easily prove (4.4) this way.

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